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Large-order asymptotes for dynamic models

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Abstract

Large-order asymptotics in dynamic field theories constructed from Langevin equations with the aid of the Martin–Siggia–Rose formalism are considered. The existence of instantons in dynamic models is discussed. Specific features of the instanton approach in the dynamic models are shown in the examples of the standard dynamic ϕ^4-based models from A to H in the common classification and Kraichnan model of passive scalar advection in turbulent flow. The results obtained demonstrate that the series of the perturbation expansions for the dynamic ϕ^4-related models is—as usual—asymptotic with zero radius of convergence. Main parameters of large-order asymptotes are determined. In the Kraichnan model, however, the situation is different. Our results show that the series here has a finite radius of convergence. This radius as well as the character of the singularity of the functions investigated was determined.

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1. Introduction

The renormalization group can be considered the most powerful method of investigation of critical and scaling behaviour. However, it produces results in a form of the sum of some, usually asymptotic, expansion (e.g. $4 - \varepsilon$) and only a few first terms are known analytically. To obtain reliable results different resummation techniques (Borel transformation, variational approach, Pade approximation and so on) have been developed. Knowledge of large-order asymptotic behaviour of perturbation series of field-theoretic models is the basis of the resummation. This behaviour has been investigated with the aid of instanton analysis which consists of generalizing the saddle-point method for the path integral and applied to the resummation problem in the prototypical static ϕ^4 model [1]. The instanton approach has been applied to a set of static quantum-field models and the behaviour of perturbation series for all essential models (ϕ^{2k}, QED, QCD, . . .) has been investigated [1] but similar information for dynamic models was not known. Fourth order of perturbation expansion of dynamic critical indices is in calculation now. Thus, the resummation problem arises here as well and demands information about the large-order asymptotics in dynamic field theories.

Let us recall the main features of the instanton approach in the example of the static theory with the action

$$S = \frac{1}{2}\partial\phi\partial\phi + \frac{g}{4!}\phi^4, \tag{1}$$

where $\phi(\mathbf{x})$ is the basic field, g is the coupling constant, all necessary integrations in coordinates (and times in dynamic models) and summation in indices of fields and partial derivatives are implied here and henceforth.

We shall use the notation $X^{[N]}$ for the Nth order contribution to the perturbation expansion in some parameter of an arbitrary quantity X. The large-order asymptote for the expansion in g of the k-point correlation function in model (1) can be determined using the expression

$$G_k^{[N]}(\mathbf{x_1}\ldots\mathbf{x_k}) = \frac{1}{2\pi \mathrm{i}}\oint\frac{\mathrm{d}g}{g}\frac{\int\mathcal{D}\phi\phi(\mathbf{x_1})\ldots\phi(\mathbf{x_k})\,\mathrm{e}^{-S-N\lg g}}{\int\mathcal{D}\phi\,\mathrm{e}^{-S}} \tag{2}$$

based on the Cauchy residue theorem. The instanton approach is the steepest-descent calculation of integrals in ϕ and g in expression (2) at large N. This leads to the stationarity equations

$$\begin{aligned}\frac{\delta S}{\delta\phi} = 0 \quad &\Rightarrow \quad -\Delta\phi + \frac{g}{6}\phi^3 = 0,\\ -\frac{\partial S}{\partial g} = \frac{N}{g} \quad &\Rightarrow \quad \frac{g}{4!}\int \mathrm{d}\mathbf{x}\,\phi^4(\mathbf{x}) = -N.\end{aligned} \tag{3}$$

The stationary solution (instanton) was found in space dimension $d = 4$ in [2]

$$\phi_{\mathrm{st}} = \frac{\sqrt{3N}}{\pi y(x-x_0)^2/y^2+1}, \qquad g_{\mathrm{st}} = \frac{1}{16\pi^2}$$

with arbitrary $\mathbf{x}_0$, y. To deal with this arbitrariness the Faddeev–Popov unit decomposition

$$\begin{aligned}1 = \int \mathrm{d}^D\mathbf{x}_0\int_{-\infty}^{+\infty}\mathrm{d}\ln y^2\delta\left[-\frac{g}{24}\int\mathrm{d}\mathbf{x}\,\phi^4(\mathbf{x})\ln\left(\frac{\mathbf{x}-\mathbf{x}_0}{y}\right)^2\right]\\ \times\,\delta^D\left[-\frac{g}{24}\int\mathrm{d}\mathbf{x}\,\phi^4(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)\right]\left[-\frac{g}{24}\int\mathrm{d}\mathbf{x}\,\phi^4(\mathbf{x})\right]^{D+1}\end{aligned}$$

was used [2].

As a result it was concluded that the large-order asymptote for an arbitrary quantity F has the form

$$F^{[N]} = N!Ca^N N^b,$$

where the constant a does not depend on the quantity F, in contrast with the constant b and the constant or function C.

In this paper, results of investigation of large-order asymptotes in dynamic models are reviewed and summarized. This paper is organized as follows. The general form of Langevin equations, the MSR formalism, models with Gibbsian static limit and the Kraichnan model of turbulent advection of passive scalar are described in section 2. The existence (non-existence) of the instanton in the dynamic models is discussed in section 3. Instanton solutions for dynamic models from A to H are presented in section 4. Instanton solution for the Kraichnan model is outlined in section 5. Section 6 contains conclusions.

2. Dynamic field theory

Considering the dynamic theory we mean the model based on the Langevin equation

$$\frac{\partial\varphi}{\partial t} + V(\varphi) = \xi, \qquad \varphi(t_0) = 0,$$

where φ is a basic field or a set of fields, $V(\varphi)$ contains an interaction, ξ is a Gaussian random field with the known correlator

$$\langle\xi(t,\mathbf{x})\xi(t',\mathbf{x}')\rangle = D(\mathbf{x}-\mathbf{x}', t-t').$$

The explicit form of the function D depends on the model considered.

This problem can be formulated as a quantum-field model using the MSR formalism [3]. In this case, the correlation and response functions can be written as

$$\frac{\int \mathcal{D}\varphi\mathcal{D}\varphi' \det M\varphi(\mathbf{x_1}, t_1)\ldots\varphi'(\mathbf{x_k}, t_k)\,\mathrm{e}^{-\overline{S}}}{\int \mathcal{D}\varphi\mathcal{D}\varphi' \det M_0\,\mathrm{e}^{-\overline{S}_0}}, \tag{4}$$

where the dynamic action has the form

$$\overline{S} = \tfrac{1}{2}\varphi' D\varphi' + \varphi'(\partial_t\varphi + V(\varphi)),$$

and the operator M is defined as

$$M = \frac{\partial}{\partial t} + \frac{\delta V}{\delta\varphi}.$$

$\overline{S}_0$ and M_0 in (4) are free parts of the dynamic action $\overline{S}$ and the operator M respectively.

An example of a dynamic model is the Kraichnan model. The advection of the passive scalar field $\varphi(\mathbf{x}, t)$ is described by the stochastic equation

$$\frac{\partial\varphi}{\partial t} - \nu\Delta\varphi + g\partial_i(\mathbf{v}_i\varphi) = \xi(\mathbf{x}, t), \tag{5}$$

where the force ξ and the velocity field $\mathbf{v}(\mathbf{x}, t)$ are Gaussian random fields with the correlators D_ξ, $\mathbf{D}_v$, the latter one being δ correlated in time and correspond to the theory of developed turbulence [4].

The other examples of a dynamic theory are the standard models of equilibrium critical dynamics, i.e. models with Gibbsian static limit. They may be described by the Langevin equation

$$\frac{\partial\varphi_a}{\partial t} + (\alpha_{ab} + \beta_{ab})\frac{\delta S}{\delta\varphi_b} = \xi_a. \tag{6}$$

Here φ_a is a set of fields, index a characterizes the field type, the matrix α can contain a linear differential operator in coordinates and the matrix β can depend on fields. ξ_a is a Gaussian random field with the following correlator:

$$\langle\xi_a(t,\mathbf{x})\xi_b(t',\mathbf{x}')\rangle = 2\alpha_{ab}\delta(t-t')\delta(\mathbf{x}-\mathbf{x}').$$

The notation of [5] has been used here. To maintain the fluctuation–dissipation theorem the parameters of (6) obey the conditions

$$\alpha^\top = \alpha, \qquad \beta^\top = -\beta, \qquad \frac{\delta\beta_{ab}}{\delta\varphi_b} = 0.$$

The static action S contains as a contribution the action of the massless ϕ^4 model (1).

Thus, the dynamic action for models A–H has the form

$$\overline{S} = -\varphi'\alpha\varphi' + \varphi'\left[\frac{\partial\varphi}{\partial t} + (\alpha+\beta)\frac{\delta S}{\delta\varphi}\right]. \tag{7}$$

Let us list the static actions S and the parameters α and β for models A–H:

Model A:

$$S = \frac{1}{2}\nabla\phi\nabla\phi + \frac{g}{4!}\phi^4, \qquad \alpha = D/2,$$

where the correlator D is a constant.

Model B:

$$S = \frac{1}{2}\nabla\phi\nabla\phi + \frac{g}{4!}\phi^4, \qquad \alpha = -\lambda\nabla^2,$$

where λ is a constant.

Model C:

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{m^2}{2} + \frac{1}{2}v_2 m\phi^2, \qquad \alpha = \begin{pmatrix} \Gamma & 0 \\ 0 & -\lambda\nabla^2 \end{pmatrix},$$

where m is an additional scalar field; Γ, λ are constants; v_2 is an additional coupling constant.

Model D:

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{m^2}{2} + \frac{1}{2}v_2 m\phi^2, \qquad \alpha = \begin{pmatrix} -\lambda\nabla^2 & 0 \\ 0 & -\lambda_1\nabla^2 \end{pmatrix},$$

where λ and λ_1 are constants.

Model F:

$$S = |\nabla\psi|^2 + \frac{g}{6}|\psi|^4 + \frac{m^2}{2} + v_2 m|\psi|^2,$$

$$\alpha = \begin{pmatrix} 0 & \lambda_\psi & 0 \\ \lambda_\psi & 0 & 0 \\ 0 & 0 & -\lambda_m\nabla^2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & \mathrm{i}v_3 & \mathrm{i}v_4\psi \\ -\mathrm{i}v_3 & 0 & -\mathrm{i}v_4\psi^* \\ -\mathrm{i}v_4\psi^* & \mathrm{i}v_4\psi & 0 \end{pmatrix},$$

where ψ is a complex-valued field; λ_ψ, λ_m are constants; v_i $(i = 2, 3, 4)$ are additional coupling constants.

Model E is F model with $v_2 = v_3 = 0$.

Model G: There are two real vector fields forming the field φ by prescription: $\varphi_a = \phi_a$ and $\varphi_{3+a} = m_a$, where $a = 1, 2, 3$. Then

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{m^2}{2}, \qquad \alpha = \begin{pmatrix} \lambda_\phi & 0 \\ 0 & -\lambda_m\nabla^2 \end{pmatrix},$$

$$\beta_{ab} = 0, \qquad \beta_{a3+b} = v_2\epsilon_{abc}\phi_c, \qquad \beta_{3+a3+b} = v_2\epsilon_{abc}m_c,$$

$a, b, c = 1, 2, 3$.

Model H:

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{c}{2}\mathbf{v}_\perp^2,$$

$$\alpha = \begin{pmatrix} -\lambda_\phi\nabla^2 & 0 \\ 0 & -\lambda_v\nabla^2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & v_2\overrightarrow{\nabla}\phi \\ -v_2\overleftarrow{\nabla}\phi & 0 \end{pmatrix},$$

where the transverse projection operator for the vector field $\mathbf{v}$ is implied; $\lambda_\psi, \lambda_m, v_2$ are constants.

3. Existence of an instanton

A natural question to ask is why the problem of large-order asymptotes was not solved earlier in dynamic models? The difficulty lies in establishing the existence of a solution of a stationarity equation similar to (3) of the static case. No general theorem is known about the existence of a global solution of the corresponding nonlinear partial differential equation. However, in [6] an approach was proposed to prove the non-existence of a solution of the stationarity equation. It is based on the use of scaling properties of the model. Let us illustrate this approach in the example of the dynamic model A.

The dynamic action has the form

$$\overline{S} = -\frac{1}{2}\phi' D\phi' + \phi'\left[\frac{\partial\phi}{\partial t} + \frac{D}{2}\left(-\nabla^2\phi + \frac{g}{6}\phi^3\right)\right].$$

Hence, the stationarity equations are of the form

$$\frac{\delta\overline{S}}{\delta\phi'} = 0 \Rightarrow -D\phi' + \frac{\partial\phi}{\partial t} + \frac{D}{2}\left[-\nabla^2\phi + \frac{g}{6}\phi^3\right] = 0,$$

$$\frac{\delta\overline{S}}{\delta\phi} = 0 \Rightarrow -\frac{\partial\phi}{\partial t} + \frac{D}{2}\left[-\nabla^2\phi' + \frac{g}{2}\phi^2\phi'\right] = 0.$$

To answer the question 'does the non-zero solution exist?' let us

(i) consider the obvious equations

$$\int \mathrm{d}\mathbf{x}\,\mathrm{d}t\,\phi'\frac{\delta\overline{S}}{\delta\phi'} = 0, \qquad \int \mathrm{d}\mathbf{x}\,\mathrm{d}t\,\phi\frac{\delta\overline{S}}{\delta\phi} = 0,$$

(ii) change variables $\phi(t, \mathbf{x}) \to \phi(\lambda' t, \lambda\mathbf{x})$

then $S = S(\lambda', \lambda)$ and we have two additional scaling equations $\lambda\partial_\lambda S|_{\lambda,\lambda'=1} = 0$, $\lambda'\partial_{\lambda'} S|_{\lambda,\lambda'=1} = 0$.

For decrease in space and time fields we then obtain a system of equations for the functionals $F_1 = \int \mathrm{d}\mathbf{x}\,\mathrm{d}t\,(\phi')^2$, $F_2 = \int \mathrm{d}\mathbf{x}\,\mathrm{d}t\,\phi'\partial_t\phi$, $F_3 = \int \mathrm{d}\mathbf{x}\,\mathrm{d}t\,\phi'\nabla^2\phi$, $F_4 = \int \mathrm{d}\mathbf{x}\,\mathrm{d}t\,\phi'\phi^3$ contributing to the dynamic action in the form

$$-DF_1 + F_2 - \frac{D}{2}F_3 + \frac{Dg}{12}F_4 = 0, \qquad F_2 - \frac{D}{2}F_3 + \frac{Dg}{4}F_4 = 0,$$

$$-\frac{dD}{2}F_1 + dF_2 - (d-2)\frac{D}{2}F_3 + \frac{dDg}{12}F_4 = 0, \qquad -\frac{D}{2}F_1 - \frac{D}{2}F_3 + \frac{Dg}{12}F_4 = 0.$$

It can be proved that these equations have no non-zero solution. Thus, a non-trivial solution of the stationarity equations in dynamic models does not exist in the class of functions decreasing at $t \to \pm\infty$.

4. Instanton analysis of models A–H

When there is no stationarity point in the steepest-descent calculation of a numeric integral the boundary contribution becomes essential. But what is the boundary of the functional space in a path integral? Our statement is that in instanton analysis for the near-equilibrium dynamic models the necessary boundary contribution is produced by functions which do not decrease in the limit $t \to +\infty$.

Let us illustrate this statement by considering the steepest-descent calculation of the parametric and path integral

$$\frac{1}{2\pi\mathrm{i}}\oint \frac{\mathrm{d}g}{g}\frac{\int \mathcal{D}\varphi\mathcal{D}\varphi' \det M\varphi\ldots\varphi'\,\mathrm{e}^{-\overline{S}-N\ln g}}{\int \mathcal{D}\varphi\mathcal{D}\varphi' \det M_0\,\mathrm{e}^{-\overline{S}_0}}.$$

The stationarity equations are

$$-\frac{\partial\varphi'}{\partial t} + \varphi'(\alpha+\beta)\frac{\delta^2 S}{\delta\varphi\delta\varphi} + \varphi'\frac{\delta\beta}{\delta\varphi}\frac{\delta S}{\delta\varphi} = 0, \tag{8}$$

$$-2\alpha\varphi' + \frac{\partial\varphi}{\partial t} + (\alpha+\beta)\frac{\delta S}{\delta\varphi} = 0. \tag{9}$$

Consider the finite time interval $t \in [t_0, T]$, then an additional condition

$$\varphi'(T, \mathbf{x}) = 0 \tag{10}$$

appears.

We note that the dynamic instanton φ_d is, in particular, a non-trivial solution of the equation

$$-\frac{\partial\varphi}{\partial t} + (\alpha-\beta)\frac{\delta S}{\delta\varphi} = 0. \tag{11}$$

Thus, we can consider just one first order in time differential equation (11) instead of the two equations (8) and (9).

To explain the choice of equation (11), let us consider the asymptotic behaviour of its solution at $t \to \pm\infty$. The stationarity equation (11) has two rather obvious time-independent solutions: $\varphi = 0$ and the static instanton $\varphi_{\rm st}$. For the trivial solution $\delta^2 S_0/\delta\varphi^2$ is a positive definite operator. For the stationary instanton the operator $\left(\delta^2 S_0/\delta\varphi^2 - |g|\varphi_{\rm st}^2/2\right)$ has at least one negative eigenvalue [1] which determines the direction along which the solution approaches the stationary instanton at $t \to \infty$. Hence, the dynamic instanton behaves as

$$\lim_{t\to-\infty}\varphi_d = 0, \qquad \lim_{t\to\infty}\varphi_d = \varphi_{\rm st},$$

which is consistent with the Gibbsian limit of the dynamic model.

Let us discuss the existence of the solution of equation (11). The analysis mentioned in section 3 is not applicable here because of the absence of the decrease of φ_d at large times. But an interesting similar argument can be put forward. Relation (11) for model A has the form

$$\frac{\partial\phi}{\partial t} - \frac{D}{2}\left(-\nabla^2\phi + \frac{g}{6}\phi^3\right) = 0.$$

Consider now the equation

$$\frac{\partial}{\partial t}\int \mathrm{d}\mathbf{x}\,\phi^2 = D\left(\int \mathrm{d}\mathbf{x}\nabla\phi\nabla\phi + \frac{g}{6}\int \mathrm{d}\mathbf{x}\,\phi^4\right). \tag{12}$$

For small ϕ (e.g. at $t \to -\infty$) the right-hand side of this equation is positive which ensures growth of $\int \mathrm{d}\mathbf{x}\,\phi^2$ with increasing time. We are interested in the case of negative g. The right-hand side of (12) is equal to zero on the stationary instanton $\phi_{\rm st}$. Therefore, the solution ϕ_d tends to $\phi_{\rm st}$, when $t \to \infty$ at least in a weak sense in a L_2 space (some measure in $\int \mathrm{d}\mathbf{x}$ in (12) to ensure the convergence of the integrals large distances is implied). Note that for the negative right-hand side of (12) decrease of $\int \mathrm{d}\mathbf{x}\,\phi^2$ with the time growth follows. In such a case there is no solution with a zero initial condition.

Relation (12) together with the asymptotic analysis presented above are the arguments in favour of the existence of a solution of (11).

As a solution of the basic equation (11), it was proposed in [7, 8] to construct the usual tree-graph solution of the nonlinear equation with the given final Cauchy condition $\varphi_d(T, \mathbf{x}) = \varphi_{\rm st}(\mathbf{x})$ and the tree-graph expansion kernel

$$\frac{1}{[4\pi\Gamma(T-t)]^{d/2}}\exp\left[-\frac{(\mathbf{x}-\mathbf{x}')^2}{4\Gamma(T-t)}\right]$$

(for model A). Obviously, the tree-graph expansion is convergent with a finite radius of convergence.

Using equation (11) it is simple to express the stationary solution φ_d' via φ_d:

$$\varphi_d' = \frac{\delta S}{\delta \varphi} = (\alpha^{-1}) \left(\frac{\partial \varphi}{\partial t} + \beta \frac{\delta S}{\delta \varphi} \right).$$

As in the static model (1), equations (8)–(10) have a non-trivial solution with negative g only.

Note that the integrals in the dynamic action (7) converge, the large-time decrease of the integrand is ensured by the φ' field. It can be proved [7, 8] that the dynamic action on the dynamic instanton solution asymptotically (the initial time instant $t_0 \to -\infty$) coincides with the static action on the static instanton

$$\overline{S}(\varphi_d, \varphi_d') = S(\varphi_{\mathrm{st}}),$$

and equation $\partial_g \overline{S} = N$ leads to $g_d = g_{\mathrm{st}}$.

Thus, the exponential factor in our steepest-descent analysis of the dynamic Green function (2) as well as the pre-exponential factor asymptotically is the same as in the corresponding equilibrium static problem. Moreover, the fluctuation determinant

$$\frac{\int \mathcal{D}\delta\varphi \mathcal{D}\delta\varphi' \det M \, \mathrm{e}^{-(\delta^2 \overline{S}/\delta\{\varphi,\varphi'\}^2)\delta\{\varphi,\varphi'\}^2}}{\int \mathcal{D}\varphi \mathcal{D}\varphi' \det M_0 \, \mathrm{e}^{-\overline{S}_0}}$$

was also calculated and it was shown that it coincides with the static one as well [7, 8].

Thus, we conclude that the asymptotic properties of the dynamic model at the leading order in N are determined by the static instanton solution which leads to the factorial growth of the large-order contributions as in the static instanton analysis. Therefore, the large-order behaviour of an arbitrary quantity F (correlation or response function or critical index) may be expressed as

$$F^{[N]} = CN! a_M^N N^b, \tag{13}$$

where $F^{[N]}$ is the N th order contribution to F of the expansion in the parameter e (e is the coupling constant g or the dimensional regularization parameter ε). The most essential for resummation schemes constants a_M in expression (13) have been determined [7, 8] for all near-equilibrium models (A–H). The exponent b in the ε-expansion contribution to the dynamic index z in the $O(n)$ symmetric dynamic theories with Gibbsian static limits was determined as

$$b = 3 + \frac{n}{2}.$$

Properties of the response functions and dynamic parts of the correlation functions were discussed in [7, 8] as well.

5. Instanton solutions for Kraichnan model

In the case of the Kraichnan model (5), we have not found the main boundary contribution in the steepest-descent calculation. The reason lies in the Gaussian character of the integration in the fields φ, φ' in the model. However, a change of variables suitable for the use of the instanton analysis, namely the Lagrangian variables approach, was proposed in [9].

Let us recall the main features of the Lagrangian variables approach in the field-theoretic model of the type considered. For an arbitrary $\mathbf{v}$ field, the Green function $G(\mathbf{x},s;\mathbf{y},t;[\mathbf{v}])$ corresponding to (5) obviously satisfies

$$(\partial_s - \nu_0 \triangle) G(\mathbf{x}, s; \mathbf{y}, t) + g \partial_{\mathbf{i}} (\mathbf{v}_{\mathbf{i}}(\mathbf{x}, s) G(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}])) = \delta(\mathbf{x} - \mathbf{y}) \delta(s - t).$$

We can eliminate the δ-functions on the right-hand side by the use of the following auxiliary quantity:

$$G(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}]) = \theta(s - t) P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}]).$$

This yields

$$(\partial_s - \nu_0 \triangle_{\mathbf{x}}) P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}]) + g\partial(\mathbf{v}(\mathbf{x}, s) P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}])) = 0$$
$$P(\mathbf{x}, s = t; \mathbf{y}, t; [\mathbf{v}]) \equiv \delta^d(\mathbf{x} - \mathbf{y}).$$

The last system of equations may be considered as a Fokker–Plank equation of some different models with $P(\mathbf{x}, s; [\mathbf{v}])$ being a simultaneous distribution function. A term-by-term comparison allows us to reconstruct this model and to write down the solution for $P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}])$ explicitly:

$$P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}]) = \langle \delta^d(\mathbf{x} - \mathbf{X}(s)) \rangle_\zeta \tag{14}$$

$$\begin{aligned} &\partial_s \mathbf{X}(s) = -g\mathbf{v}(\mathbf{X}(s), s) + \zeta(s) \\ &\mathbf{X}|_{s=t} = \mathbf{y}, \qquad D_\zeta = 2\nu_0, \end{aligned} \tag{15}$$

where $\langle \cdots \rangle_\zeta$ denotes an average over the auxiliary random Gaussian force ζ (D_ζ being its correlator). Equations (14) and (15) describe particle motion in a random force field ζ. We see that the problem reduces to the study of interacting particles moving in a random medium, since all the quantities in the Kraichnan model can be expressed via the Green function $G(\mathbf{x}, s; \mathbf{y}, t)$ in the random field $\mathbf{v}$. The medium is inhomogeneous: thus, (15) would be exactly the equation for the ordinary Brownian motion were there not the term containing the velocity field with the explicit dependence on the particle position.

Let us insert in (14) an additional path integration over an auxiliary field $\mathbf{c}(s)$ having in the integrand the Dirac δ^F function with respect to the field variable $\mathbf{c}$:

$$P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}]) = \left\langle \int D\mathbf{c}\, \delta^F(\mathbf{c}(s) - \mathbf{X}(s)) \delta^d(\mathbf{c}(s) - \mathbf{x}) \right\rangle_\zeta \tag{16}$$

and use the infinite-dimensional analogue of the well-known finite dimensional identity for the δ-function:

$$\delta^F(\mathbf{c}(s) - \mathbf{X}(s)) = \det\left(\partial_s + g\frac{\delta \mathbf{v}}{\delta \mathbf{c}}\right) \delta^F(\partial_s \mathbf{c} + g\mathbf{v}(\mathbf{c}, s) - \zeta).$$

Then the d-dimensional δ^d-function in (16) can be transformed into the boundary condition for $\mathbf{c}(s)$:

$$P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}]) = \left\langle \int_{\mathbf{c}(t)=\mathbf{y}}^{\mathbf{c}(s)=\mathbf{x}} D\mathbf{c} \det\left(\partial_s + g\frac{\delta \mathbf{v}}{\delta \mathbf{c}}\right) \delta^F(\partial_s c + g\mathbf{v}(\mathbf{c}, s) - \zeta) \right\rangle_\zeta .$$

Converting the δ^F-function to the Fourier-like path-integral form we obtain

$$P(\mathbf{x}, s; \mathbf{y}, t; [\mathbf{v}]) = M \int_{\mathbf{c}(t)=\mathbf{y}}^{\mathbf{c}(s)=\mathbf{x}} D\mathbf{c} D\mathbf{c}' \det\left(\partial_s + g\frac{\delta \mathbf{v}}{\delta \mathbf{c}}\right) \exp(-\nu \mathbf{c}'^2 + \mathrm{i}\mathbf{c}'\dot{\mathbf{c}} + \mathrm{i}g\mathbf{c}'\mathbf{v}(\mathbf{c}, \tau)).$$

Here, the sum over the vector indices of $\mathbf{c}(\tau)$, $\mathbf{c}'(\tau)$ and $\mathbf{v}(\mathbf{c}, \tau)$ fields as well as the integration over the field arguments is implied. Henceforth, $\dot{\mathbf{c}}$ denotes $\partial \mathbf{c}(\tau)/\partial \tau$, and M is a normalization factor which appeared due to a functional determinant of the last δ^F-function transformation. This field theory coincides exactly with the standard MSR formalism. This is proved by comparison of the diagrammatic expansions using the proper regularization of the appearing functional determinants. The regularized $\det(\partial_s + g\delta\mathbf{v}/\delta\mathbf{c})$ can be considered as a constant independent of the fields $\mathbf{v}$, $\mathbf{c}$, like in the usual MSR formalism [10], and it can be included in the factor M. Thereupon M is determined by the free theory (at $g = 0$) and is restored by comparison with it:

$$\langle\varphi(\mathbf{x}, t_0)\varphi'(\mathbf{y}, t)\rangle\bigg|_{g=0} = M\int_{\mathbf{c}(t)=\mathbf{y}}^{\mathbf{c}(t_0)=\mathbf{x}} D\mathbf{c}D\mathbf{c}'\exp\left(-\nu\mathbf{c}'^2 + \mathrm{i}\mathbf{c}'\dot{\mathbf{c}}\right)$$

$$= \frac{1}{(4\pi\nu T)^{d/2}}\exp\left(-\frac{(\mathbf{x}-\mathbf{y})^2}{4\nu T}\right), \qquad T \equiv t_0 - t.$$

The objects of investigation in the Kraichnan model are the single-time structure functions:

$$\langle[\varphi(t,\mathbf{x}) - \varphi(t,\mathbf{x}')]^n\rangle,$$

whose long-range behaviour is determined by the properties of the composite operators φ^n [10]. To investigate renormalization of these operators using Lagrangian variables, consider the theory with the action

$$S = N\left\{\nu\sum_{i=1}^{n}\mathbf{c}_i'^2 - \mathrm{i}\sum_{i=1}^{n}\mathbf{c}_i'\dot{\mathbf{c}}_i + \frac{g\nu}{2}\sum_{i\neq j}\mathbf{c}_i' D(\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_j'\right\},$$

where $\mathbf{c}_i$ is the position of the ith 'liquid particle' and $\mathbf{c}_i'$ its momentum, D is a random velocity correlator and g is a coupling constant.

The stationarity equations in these variables have a form

$$-\mathrm{i}\dot{\mathbf{c}}_m' = u\eta\sum_{\substack{l\\ l\neq m}}\mathbf{c}_m'\frac{\partial D(\mathbf{c}_m - \mathbf{c}_l)}{\partial(\mathbf{c}_m - \mathbf{c}_l)}\mathbf{c}_l',$$

$$\mathrm{i}\dot{\mathbf{c}}_m = u\eta\sum_{\substack{l\\ l\neq m}}D(\mathbf{c}_m - \mathbf{c}_l)\mathbf{c}_l' + 2\eta\mathbf{c}_m'.$$

These equations have the solution

$$\mathbf{p}(\tau) = \frac{\mathrm{i}\dot{\mathbf{q}}(\tau)}{2\nu - g\nu D_v(\mathbf{q}(\tau))}, \qquad \dot{\mathbf{q}}(\tau) = \frac{I_1(\mathbf{x})}{T}\sqrt{2\nu - g\nu D_v(\mathbf{q}(\tau))},$$

$$I_1(\mathbf{x}) = \int_0^x\frac{\mathrm{d}z}{\sqrt{2\nu - g\nu D_v(z)}},$$

[9] where $\mathbf{p} = \mathbf{c}_1' - \mathbf{c}_2'$, $\mathbf{q} = \mathbf{c}_1 - \mathbf{c}_2$, which were applied to the problem of investigation of the large-order asymptotes in the Kraichnan model.

As a result, for the ϵ expansion of anomalous dimensions of the composite operators (see [10])

$$\gamma_{\varphi^n} = \sum_{N\geqslant 0}\gamma_{\varphi_n}^{(N)}\epsilon^N,$$

the large-order asymptote was obtained [11] in the form

$$\gamma_{\varphi^n}^{(N)} \sim \left[\frac{-2\alpha}{(d-1+\alpha)}\right]^N,$$

which determines the radius of convergence of the ϵ expansion

$$\epsilon_c = -\frac{d-1+\alpha}{2\alpha}.$$

This information was used [11] for resummation of the γ_{φ^n} series: we extract the singularity of the expansion adopting the simple rational representation

$$\gamma_{\varphi^n} = \sum_{k=1}^{\infty} \gamma_{\varphi^n}^{(k)} \epsilon^k = \frac{\sum_{k=1}^{\infty} \widetilde{\gamma}_{\varphi^n}^{(k)} \epsilon^k}{\epsilon - \epsilon_c}.$$

The results obtained were confirmed by comparison with the exact solution for the composite operator ϕ^2 [10].

6. Conclusions

The large-order asymptotic analysis of perturbation expansions in dynamic models is more difficult problem than in static models, mainly because of the absence of a solution of the stationarity equation in the natural function class. Nevertheless, the instanton method is applicable here after the appropriate choice of the function class (with non-trivial boundary conditions) in the case of the near-equilibrium models of critical dynamics or after the choice of functional variables with a non-trivial behaviour at the boundary in the case of the Kraichnan model.

The results of the large-order asymptotic analysis in dynamic models demonstrate a great variety compared with statics. It seems that the general situation in the dynamic models is the factorial growth of the large-order coefficients of the perturbation expansion. Nevertheless, convergent perturbation series in nonlinear dynamic models is also possible, as in the Kraichnan model.

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